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Systems of conservation laws in the setting of the projective theory of congruences: reducible and linearly degenerate systems

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Abstract

We review some of the recent results in the projective-geometric theory of systems of conservation laws with emphasis on linearly degenerate systems, reducible systems and systems of Temple's class, the equations of associativity of two-dimensional topological field theory being the main example. Our construction reveals a close relationship of these classes of systems with linear congruences and linear complexes of lines in projective space.

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1. Introduction

Hyperbolic systems of conservation laws are quasilinear PDEs of the form

$$u^i_t = f^i(u)_x = v^i_j(u)u^j_x, \quad v^i_j = \frac{\partial f^i}{\partial u^j}, \quad i = 1, \ldots, n. \tag{1}$$

They naturally arise in a variety of physical applications and are known to possess a rich mathematical and geometric structure [9,16,17,24,26]. It was observed recently that many constructions of the theory of systems of conservation laws are, in a sense, parallel to that of the projective theory of congruences.

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The congruence of lines, associated with the system (1) (see [2] and [3]), is an n-parameter family of lines

$$y^i = u^i y^0 - f^i(u), \quad i = 1, \ldots, n, \tag{2}$$

in $(n+1)$-dimensional projective space P^{n+1} with affine coordinates $y^0, \ldots, y^n$. In the case $n = 2$ this is a two-parameter family, or a congruence of lines in P^3. In the 19th century the theory of congruences was one of the most popular chapters of classical differential geometry (see, e.g., [15]). We keep the name "congruence" for any n-parameter family of lines (2) in P^{n+1}.

It turns out that the basic concepts of the theory of systems of conservation laws, such as shock and rarefaction curves, Riemann invariants, reciprocal transformations, linearly degenerate systems, etc. acquire a clear and simple projective interpretation when reformulated in the language of the line geometry. For instance, this correspondence enabled the classification of systems of Temple class [27] to be reduced to a much simpler geometric problem of the classification of congruences with either planar or conical developable surfaces. In particular, some of the results of [27] became intuitive geometric statements about families of lines in projective space.

Let us briefly review some of the necessary material from [2,3,27]. Consider the eigenvalues $\lambda^i(u)$ of the matrix $v^i_j(u)$ (called the characteristic speeds of system (1)), assumed real and pairwise distinct. Let $\boldsymbol{\xi}_i(u)$ be the corresponding eigenvectors: $v\boldsymbol{\xi}_i = \lambda^i \boldsymbol{\xi}_i$. We denote by L_i the Lie derivative in the direction of $\boldsymbol{\xi}_i$ and introduce the expansions

$$[L_i, L_j] = c^k_{ij} L_k. \tag{3}$$

Rarefaction curves, defined as the integral curves of the eigenvectors $\boldsymbol{\xi}_i$, play a crucial role in the theory of hydrodynamic type systems. Thus, there are n families of rarefaction curves, and for any point in u-space there is exactly one rarefaction curve from each family passing through it. Due to the correspondence (2), a curve in u-space defines a ruled surface, i.e., a one-parameter family of lines in P^{n+1}. In [2] the following important property was established: *Ruled surfaces defined by rarefaction curves of the ith family are developable, i.e., their rectilinear generators are tangential to a curve. This curve can be parametrized in the form*

$$y^0 = \lambda^i, \; y^1 = u^1\lambda^i - f^1(u), \ldots, y^n = u^n\lambda^i - f^n(u), \tag{4}$$

where u varies along the corresponding rarefaction curve.

The curve (4) constitutes a singular locus of the developable surface called its *cuspidal edge*.

The focal hypersurface $M_i \subset P^{n+1}$ is a collection of cuspidal edges corresponding to rarefaction curves of the ith family. Therefore, parametric equations of M_i coincide with (4), where u is now allowed to take all possible values. By a construction, each line of the congruence (2) is tangential to M_i. The idea of focal hypersurfaces is obviously borrowed from optics: thinking of the lines of a congruence as the rays of light, one can intuitively imagine focal hypersurfaces as the locus in P^{n+1} where the light concentrates (this explains why in the German literature focal hypersurfaces are called 'Brennflächen', which can be translated as 'burning surfaces'). Since the system of conservation laws (1) is strictly hyperbolic, there are precisely n developable surfaces passing through a line of the congruence (2), and each line is tangential to n focal hypersurfaces.

Shock curves play a fundamental role in the theory of weak solutions of systems (1). A shock curve with the vertex at u_0 is the set of points in u-space such that

$$\sigma\left(u^i - u^i_0\right) + f^i(u) - f^i(u_0) = 0, \quad i = 1, \ldots, n, \tag{5}$$

for some function $\sigma(u, u_0)$. For any u on the shock curve the discontinuous function

$$u(x,t) = u_0, \quad x < \sigma t,$$
$$u(x,t) = u, \quad x > \sigma t,$$

is a weak solution of (1). Notice that (5) implies that the lines l_u and l_{u_0}, corresponding to the points u and u_0, intersect in P^{n+1}. This implies that the shock curve with the vertex at u_0 defines a special ruled surface of the congruence (2) consisting of the lines of the congruence intersecting l_{u_0} [2]. Lax showed that a shock curve with the vertex at a generic point u_0 splits into n branches, the ith branch being C^2-tangent of the associated rarefaction curve of the ith family passing through u_0. As pointed out by a number of authors, there are situations when shock curves coincide with their associated rarefaction curves. Systems with coinciding shock and rarefaction curves were studied by Temple [27]. His main theorem can be formulated as follows.

Rarefaction curves of the ith family coincide with the associated branches of the shock curve if and only if either

1) *every rarefaction curve of the ith family is a straight line in the u-space or*
2) *the characteristic velocity λ^i is constant along rarefaction curves of the ith family,*

$$L_i(\lambda^i) = 0.$$

This theorem introduces two natural classes of systems:

Systems with linear rarefaction curves, geometrically characterized by the planarity of cusp-edges (or, equivalently, planarity of developable surfaces) of the associated congruence (1) (see [2]), and

Linearly degenerate systems, characterized by the condition 2) being satisfied for all $i = 1, \ldots, n$. Geometrically, this condition means that developable surfaces of the congruence (1) are conical, and therefore all focal hypersurfaces M_i degenerate into submanifolds of codimension two [2]. This geometric result allows one to write down a general implicit formula for the fluxes $f_i(u)$ of linearly degenerate systems as shown in Section 2 (these formulae were previously known for $n = 2$ only). As demonstrated in [2] and [3], the properties formulated above provide an intuitive geometric proof of Temple's Theorem.

Reducible systems of conservation laws, which have been a subject of research in [1] and [6], are investigated and characterized geometrically in Section 3. They can be defined as systems transformable to nth order scalar PDE

$$f\left(\frac{\partial^n u}{\partial x^n}, \frac{\partial^n u}{\partial x^{n-1}\partial t}, \ldots, \frac{\partial^n u}{\partial t^n}\right) = 0$$

(see Section 3 for details). In geometric language, the reducibility of system (1) implies that the associated congruence (2) belongs to the intersection of special linear complexes of rank 4.

T-systems, considered in Section 4, simultaneously satisfy both conditions of Temple's Theorem, namely, all rarefaction curves are rectilinear (in u-space), and all eigenvalues λ^i are linearly degenerate. Systems of this type naturally arise in the theory of equations of associativity of 2D topological field theory [8]. In view of the results formulated above, developable surfaces of the corresponding congruence (2) must be planar and conical simultaneously, and therefore are planar pencils of lines. The corresponding focal hypersurfaces M_i degenerate into n submanifolds of codimension 2. In all examples discussed in Section 4, the focal submanifolds M_i are glued together to form an algebraic variety $V^{n-1} \subset P^{n+1}$ of codimension 2, so that the lines of the congruence (2) can be characterized as n-secants of V^{n-1}. In Section 4 we propose a complete description of T-systems for $n=2$ (which is elementary) and $n=3$ (which is related to the classical result of Castelnuovo [7] classifying linear congruences in P^4). For $n \geqslant 4$ we have no classification so far. Examples of n-component T-systems include systems of conservation laws related to linear congruences in P^{n+1} (Section 5) and a class of completely exceptional equations of Monge–Ampère type (Section 6).

Riemann invariants. We say that system (1) possesses Riemann invariants if one can find new dependent variables $R^i(u)$ such that Eqs. (1) take the diagonal form

$$R^i_t = \lambda^i R^i_x, \quad i = 1, \ldots, n,$$

(no summation). Analytically, this amounts to requiring $c^i_{jk} = 0$ for any $i \neq j \neq k$ in Eqs. (3). As shown in [3], the existence of Riemann invariants implies that the focal nets (i.e., nets cut out by developable surfaces on each of the focal submanifolds M_i) are conjugate and holonomic. Congruences of this type and their transformations have been a subject of extensive research in projective differential geometry. Among others, the Laplace and Lévy transformations play fundamental roles. Being translated into the language of systems of conservation laws, these constructions lead to nontrivial transformations of semi-Hamiltonian systems of hydrodynamic type which were investigated recently in [10] and [12].

Reciprocal transformations. Let $B(u)\,dx + A(u)\,dt$ and $N(u)\,dx + M(u)\,dt$ be two conservation laws of system (1), understood as the one-forms which are closed by virtue of (1). In the new independent variables X, T defined by

$$dX = B(u)\,dx + A(u)\,dt, \qquad dT = N(u)\,dx + M(u)\,dt, \tag{6}$$

system (1) takes the form

$$U^i_T = F^i(U)_X, \quad i = 1, \ldots, n, \tag{7}$$

where

$$U^i = \frac{u^i M - f^i N}{BM - AN}, \qquad F^i = \frac{f^i B - u^i A}{BM - AN}.$$

The transformations (6) are called *reciprocal*. Reciprocal transformations are known to preserve linear degeneracy (see [11]). Moreover, particular reciprocal transformations (6) with both integrals (6) being linear combinations of the canonical integrals $u^i\,dx + f^i\,dt$ defining system (1),

$$\begin{aligned} dX &= \bigl(\alpha_i u^i + \alpha\bigr)\,dx + \bigl(\alpha_i f^i + \tilde{\alpha}\bigr)\,dt, \\ dT &= \bigl(\beta_i u^i + \beta\bigr)\,dx + \bigl(\beta_i f^i + \tilde{\beta}\bigr)\,dt, \end{aligned} \tag{8}$$

(here $\alpha_i, \alpha, \tilde{\alpha}, \beta_i, \beta, \tilde{\beta}$ are arbitrary constants) are known to preserve the class of T-systems [2]. It was pointed out in [2] that the transformation group generated by reciprocal transformations (8) and affine changes of variables u^i, is isomorphic to the group of projective transformations of P^{n+1}. Thus, the classification of systems of conservation laws up to transformations (8) and affine changes of u's is equivalent to the classification of the corresponding congruences up to projective equivalence. Actually, this observation was the main motivation for introducing the above geometric correspondence.

2. Linearly degenerate systems: implicit formulae for the fluxes

Let us state once again the following result from [2]:

Theorem 1. *The characteristic velocity λ^i is linearly degenerate (that is, $L_i\lambda^i = 0$) if and only if developable surfaces of the ith family are conical, that is, their generators meet in a point. The corresponding focal hypersurface M_i degenerates into a submanifold of codimension two.*

If all eigenvalues are linearly degenerate, the corresponding congruence will consist of n-secants of n fixed submanifolds of codimension two in P^{n+1}. Conversely, given n arbitrary submanifolds of codimension two in P^{n+1}, the manifold of n-secant lines thereof is n-dimensional and therefore defines (locally) a linearly degenerate system of conservation laws, after we parametrize the lines in the form (2). Notice that the condition of linear degeneracy leads to a system of n second order nonlinear PDEs for the fluxes $f^i(u)$. In the case $n = 2$ this system is

$$\begin{aligned}
&\left(f_2^2 - f_1^1\right) f_{11}^1 - 2 f_1^2 f_{12}^1 - f_1^2 f_{22}^2 - f_2^1 f_{11}^2 = 0,\\
&\left(f_1^1 - f_2^2\right) f_{22}^2 - 2 f_2^1 f_{12}^2 - f_2^1 f_{11}^1 - f_1^2 f_{22}^1 = 0,
\end{aligned} \tag{9}$$

where $f_1^1 = \partial f^1/\partial u^1$, $f_2^1 = \partial f^1/\partial u^2$, etc. A general integral of this system is given by the implicit formulae

$$\begin{aligned}
u^1 &= \frac{g^1(p) - g^2(q)}{q - p}, \qquad & u^2 &= \frac{h^1(p) - h^2(q)}{q - p},\\
f^1 &= \frac{q g^1(p) - p g^2(q)}{q - p}, \qquad & f^2 &= \frac{q h^1(p) - p h^2(q)}{q - p},
\end{aligned} \tag{10}$$

where $g^1(p)$, $h^1(p)$, $g^2(q)$, $h^2(q)$ are four arbitrary functions of the arguments indicated. Although for $n > 2$ system (9) becomes more complicated, one can write down its general integral in a similar implicit form as follows. Let us parametrize the first focal submanifold M_1 in the form

$$\begin{aligned}
y^{n-1} &= g\left(y^0, y^1, \ldots, y^{n-2}\right),\\
y^n &= h\left(y^0, y^1, \ldots, y^{n-2}\right),
\end{aligned}$$

where g and h are two arbitrary functions. The condition saying that the line (2) meets M_1, takes the form

$$\begin{aligned}
u^{n-1} y^0 - f^{n-1} &= g\left(y^0, u^1 y^0 - f^1, \ldots, u^{n-2} y^0 - f^{n-2}\right),\\
u^n y^0 - f^n &= h\left(y^0, u^1 y^0 - f^1, \ldots, u^{n-2} y^0 - f^{n-2}\right).
\end{aligned}$$

Excluding y^0 from these two equations, we obtain one relation among $u^1, \ldots, u^n$ and $f^1, \ldots, f^n$. Doing this for each of the focal submanifolds M_i, we end up with a system of n equations which, implicitly, define f's as functions of u's. Notice that $2n$ functions of $n-1$ variables (on which a general solution of the system of second order PDEs for $f^1, \ldots, f^n$ should depend) enter explicitly into the answer.

3. Reducible systems and linear complexes

Let us consider a PDE of the form

$$\frac{\partial^n u}{\partial t^n}=f\left(\frac{\partial^n u}{\partial x^n},\frac{\partial^n u}{\partial x^{n-1}\partial t},\dots,\frac{\partial^n u}{\partial x\partial t^{n-1}}\right).$$

Introducing the variables $a^1=\frac{\partial^n u}{\partial x^n}$, $a^2=\frac{\partial^n u}{\partial x^{n-1}\partial t},\dots,a^n=\frac{\partial^n u}{\partial x\partial t^{n-1}}$, we can rewrite it as a system of conservation laws

$$a^1_t=a^2_x,\ a^2_t=a^3_x,\dots,a^n_t=f\left(a^1,a^2,\dots,a^n\right)_x. \tag{11}$$

Definition 1. A system of conservation laws is said to be *reducible* if it can be cast into the form (11) by an appropriate reciprocal transformation (8) together with an affine change of dependent variables.

Notice that all examples discussed so far are reducible by a construction. There exists a simple geometric criterion for a system of conservation laws to be reducible.

Definition 2. A linear complex in P^{n+1} is a family of lines whose Plücker coordinates P^{ij} are subject to a linear constraint $A_{ji}P^{ij}=0$, where $A_{ij}=-A_{ji}=\text{const}$.

Recall that to the line in P^{n+1} passing through the points with homogeneous coordinates $X=(X^0:X^1:\dots:X^{n+1})$ and $Y=(Y^0:Y^1:\dots:Y^{n+1})$ there corresponds a point in the Grassmanian $\mathbf{G}(1,n+1)$ with Plücker coordinates $P^{ij}=X^iY^j-X^jY^i$, $i,j=0,\dots,n+1$. If one considers Plücker coordinates as an $(n+2)\times(n+2)$ skew-symmetric matrix P of rank 2, any linear constraint can be rewritten in the form $\operatorname{tr}AP=0$, where A is a skew-symmetric matrix. Intersection of $n-1$ linear complexes is given by $n-1$ linear equations

$$\operatorname{tr}A^\alpha P=0,\quad \alpha=1,\dots,n-1, \tag{12}$$

where the matrices A^α are linearly independent. Define the map $A(\mu)$ by

$$CP^{n-2}\ni(\mu_1:\mu_2:\dots:\mu_{n-1})\to A(\mu)=\sum_\alpha\mu_\alpha A^\alpha.$$

Theorem 2. *A system of conservation laws* (1) *is reducible iff the corresponding congruence* (2) *lies in the intersection of* $n-1$ *linear complexes* (12) *such that*

1) $\operatorname{rank}A(\mu)=4$ *for all* $\mu\in CP^{n-2}$,

2) *there exists an* n*-dimensional linear subspace* $L\subset V^{n+2}$*, which is Lagrangian with respect to all skew-symmetric scalar products* $\{X,Y\}_\mu:=X^{\mathrm{T}}A(\mu)Y$.

Recall that the subspace of a linear space with a skew-symmetric scalar product $\{,\}$ is called Lagrangian if $\{X,Y\}=0$ for all $X,Y\in L$. In the parametrization (2), the Plücker coordinates of a congruence in P^{n+1} are

$$1,\qquad u,\qquad f,\qquad u\wedge f,$$

so that the corresponding matrix P is

$$P=\begin{pmatrix} 0 & 1 & u^1 & \dots & u^n \\ -1 & 0 & f^1 & \dots & f^n \\ -u^1 & -f^1 & & & \\ \vdots & \vdots & & u^i f^j - u^j f^i & \\ -u^n & -f^n & & & \end{pmatrix},$$

Eqs. (11) imply that the basis of matrices A^α can be chosen in the form

$$A^\alpha=\begin{pmatrix} 0 & D^\alpha \\ -(D^\alpha)^{\mathrm T} & 0 \end{pmatrix}$$

where D^α are $2\times n$ matrices with only two nonzero entries:

$$D^1=\begin{pmatrix} 0 & 1 & 0 & \dots & 0 \\ -1 & 0 & 0 & \dots & 0 \end{pmatrix},\dots,D^{n-1}=\begin{pmatrix} 0 & \dots & 0 & 0 & 1 \\ 0 & \dots & 0 & -1 & 0 \end{pmatrix}.$$

Notice that the condition $\operatorname{tr} A^\alpha P=0$ is equivalent to $f^\alpha=u^{\alpha+1}$. It is easy to see that both conditions 1) and 2) are fulfilled. To prove that these conditions are also sufficient is a bit more difficult. Here we restrict our consideration to the cases $n=2$, $n=3$ and give a full proof of the theorem in Appendix A.

In the case $n=2$ conditions 1) and 2) imply that the 4×4 matrix A determining the linear complex in question, is nondegenerate. Any such matrix can be transformed into the form

$$\begin{pmatrix} 0 & 0 & 0 & 1 \\ 0 & 0 & -1 & 0 \\ 0 & 1 & 0 & 0 \\ -1 & 0 & 0 & 0 \end{pmatrix}$$

with the corresponding congruence in P^3

$$Y^1=aY^0-bY^{n+1},\qquad Y^2=bY^0-f(a,b)Y^{n+1},$$

giving rise to a reducible 2-component system.

Now let the congruence in P^4 associated with a 3-component system lie in the intersection of 2 general linear complexes. If the matrix $A(\mu)$ has rank 4 for any $\mu\in CP^1$, the kernel $\xi(\mu)$ of $\{,\}_\mu$ is one-dimensional. So the map $\xi(\mu): CP^1\ni\mu\to P^4$ is correctly defined, the image of CP^1 being a plane conic (see [7]). Condition 1) guarantees that this conic is nondegenerate. As follows from the results of Castelnuovo [7], the intersection of 2 general linear complexes in P^4 is projectively equivalent to

$$y^1=ay^0-b,\qquad y^2=by^0-c,\qquad y^3=cy^0-f. \tag{13}$$

Congruences belonging to the complex (13) are specified by one extra relation of the form $f=f(a,b,c)$, thus giving rise to reducible systems.

Remark 1. The Lagrangian subspace L in Theorem 2 can be described constructively as a linear span of $\xi(\mu)$, the $\xi(\mu)\in V^{n+2}$ being the $(n-2)$-dimensional kernel of $\{X,Y\}_\mu$, so that the criterion given by Theorem 2 is effective. One has to construct the linear span L of $\xi(\mu)$ and verify two conditions: $\dim L=n$ and $\{X,Y\}_\mu=0$ for all $X,Y\in L$, $\mu\in CP^{n-2}$.

4. Classification of T-systems

In this section we classify 2- and 3-component T-systems of conservation laws. Recall that these systems satisfy the following two properties:

(a) The integral trajectories of the eigenvectors ξ^i (which are the rarefaction curves of system (1)) are straight lines in coordinates $u^1, \ldots, u^n$.

(b) The eigenvalues λ^i are constant along rarefaction curves of the ith family.

Geometrically, these two conditions imply that developable surfaces of the corresponding congruence (2) must be planar and conical simultaneously, and therefore are planar pencils of lines. The corresponding focal hypersurfaces M_i degenerate into n submanifolds of codimension 2. One can readily establish that for $n = 2$ such congruences are linear (that is, defined by two linear equations in Plücker coordinates) and consist of all lines intersecting two fixed skew lines in P^3. Since any two linear congruences in P^3 are projectively equivalent, there exists essentially a unique two-component T-system.

Example 1. Consider the wave equation

$$f_{tt} - f_{xx} = 0. \tag{14}$$

Introducing the variables $a = f_{xx},\ b = f_{xt}$, we readily rewrite (14) as a linear two-component system of conservation laws

$$a_t = b_x, \qquad b_t = a_x \tag{15}$$

which is obviously a T-system (any linear system of conservation laws is a T-system since its eigenvalues and eigenvectors are constant). The corresponding congruence (2)

$$y^1 = ay^0 - b, \qquad y^2 = by^0 - a \tag{16}$$

consists of all lines intersecting the two skew lines $y^0 = 1,\ y^1 = -y^2$ and $y^0 = -1,\ y^1 = y^2$.

Example 2. Consider the Monge–Ampère equation

$$f_{xt}^2 - f_{xx} f_{tt} = 1. \tag{17}$$

Introducing the variables $a = f_{xx},\ b = f_{xt}$ [21], we rewrite (17) as a two-component system of conservation laws

$$a_t = b_x, \qquad b_t = \left(\frac{b^2 - 1}{a}\right)_x \tag{18}$$

which also proves to be a T-system. The corresponding congruence

$$y^1 = ay^0 - b, \qquad y^2 = by^0 - \frac{b^2 - 1}{a} \tag{19}$$

consists of all lines intersecting the two skew lines $y^1 = 1,\ y^0 = y^2$ and $y^1 = -1,\ y^0 = -y^2$.

Since congruences (16) and (19) are projectively equivalent, the corresponding systems (15) and (18) are reciprocally related, thus providing a linearization of the nonlinear Monge–Ampère equation (17) (which, of course, is not a new result).

The main result of this section is the classification of three-component T-systems or, in geometric language, congruences in P^4 whose developable surfaces are planar pencils of lines. The main example which motivated our research comes from the theory of equations of associativity of two-dimensional topological field theory.

Example 3. Let us consider the Monge–Ampère type equation

$$f_{ttt} = f_{xxt}^2 - f_{xxx} f_{xtt}, \tag{20}$$

known as the WDVV or the associativity equation, which was thoroughly investigated by Dubrovin in [8]. Introducing the variables $a = f_{xxx}$, $b = f_{xxt}$, $c = f_{xtt}$ [20], we rewrite (20) as a three-component system of conservation laws

$$a_t = b_x, \qquad b_t = c_x, \qquad c_t = \left(b^2 - ac\right)_x \tag{21}$$

which was observed to be a T-system in [2]. The corresponding congruence in P^4

$$y^1 = ay^0 - b, \qquad y^2 = by^0 - c, \qquad y^3 = cy^0 - b^2 + ac \tag{22}$$

coincides with the set of trisecant lines of the Veronese variety projected from P^5 into P^4 (see the discussion below). In this sense the projected Veronese variety plays the role of the focal variety of the congruence (22). As follows from the classification result presented below, this example is, in a sense, generic.

Definition 3. A congruence (2) is called linear (or general linear) if its Plücker coordinates

$$1, \qquad u^i, \qquad f^i, \qquad u^i f^j - u^j f^i$$

satisfy n linear equations of the form

$$\alpha + \alpha_i u^i + \beta_i f^i + \alpha_{ij}\left(u^i f^j - u^j f^i\right) = 0 \tag{23}$$

where $\alpha, \alpha_i, \beta_i, \alpha_{ij}$ are arbitrary constants. Equivalently, linear congruences can be defined as intersections of n linear complexes.

Notice that Eqs. (23), being linear in f, define f^i as explicit functions of u.

Theorem 3 [4]. *Congruence corresponding to three-component T-systems are linear.*

Remark. Given a congruence in P^4 whose developable surfaces are planar pencils of lines, its Plücker image is a three-dimensional submanifold M^3 of the Grassmanian $\mathbf{G}(1,4)$ covered by a two-parameter family of lines, the images of planar pencils. Moreover, there are three lines passing through each point of M^3. Algebraic threefolds covered by lines were recently classified by Mezzetti and Portelli in [19]. It was demonstrated that threefolds with three lines through each point are intersections of $\mathbf{G}(1,4)$ with a linear subspace of codimension 3, that is, Plücker images of linear congruences. In [4] we approached the classification problem from the point of view of local differential geometry, without imposing any additional algebro-geometric restrictions. It turned out, however, that our local differential-geometric assumptions (namely, that developable surfaces are planar pencils) already imply the algebraizability. Unlike the (elementary) case of P^3, the proof of the linearity of these congruences in P^4 requires a long computation bringing a certain exterior differential system into involutive form. Notice that the linearity does not necessarily hold in P^5 as simple examples from Section 6 show. Once the linearity is established, one can make use of the results of Castelnuovo [7] who classified linear congruences in P^4. This gives a list of six three-component T-systems which are not reciprocally related. Below we list them as scalar third-order Monge–Ampère type equations. They assume the form (1) in the variables $a = f_{xxx}$, $b = f_{xxt}$, $c = f_{xtt}$. As systems of conservation laws, they differ by a number of Riemann invariants they possess. Geometrically, for T-systems, the existence of a Riemann invariant implies the reducibility of the focal variety of the corresponding congruence: if a T-system possesses k Riemann invariants, the focal variety contains k linear subspaces of codimension two [4].

Theorem 4 [4]. *Any strictly hyperbolic three-component T-system can be reduced by a reciprocal transformation to one from the following list.*

(I) *T-systems which possess no Riemann invariants,*

$$f_{xxx} f_{ttt} - f_{xxt} f_{ttx} = 1 \tag{24}$$

and

$$f_{xxt}^2 + f_{xtt}^2 - f_{xxx} f_{xtt} - f_{ttt} f_{xxt} = 1. \tag{25}$$

The focal varieties of the corresponding congruences are non-singular projections of the Veronesé variety into P^4. The congruences consist of the trisecant lines of these projections. Notice that there are two different projections which are not equivalent over the reals.

(II) *T-systems which possess one Riemann invariant,*

$$f_{xxx} f_{ttt} - f_{xxt} f_{ttx} = 0 \tag{26}$$

and

$$f_{xxt}^2 + f_{xtt}^2 - f_{xxx} f_{xtt} - f_{ttt} f_{xxt} = 0. \tag{27}$$

The corresponding focal varieties are reducible and consist of a cubic scroll and a plane which intersects the cubic scroll along its directrix. Notice that Eqs. (24) *and* (26) *are related to* (25) *and* (27) *by a complex change of variables $x \to (x+t)/\sqrt{2}$, $t \to i(x-t)/\sqrt{2}$.*

(III) *T-system with two Riemann invariants,*

$$f_{xtt}^2 - f_{xxt} f_{ttt} = 1 \tag{28}$$

which reduces to the Monge–Ampère equation (17) *for $\tilde{f} = f_t$. The corresponding focal variety consists of a two-dimensional quadric and two planes which intersect the quadric along rectilinear generators of different families.*

(IV) *T-system with three Riemann invariants,*

$$f_{ttt} - f_{xxt} = 0. \tag{29}$$

The corresponding focal variety consists of three planes.

We discuss the geometry of these examples in some more detail below.

Remark. Eq. (24) was discussed by Dubrovin in [8]. As shown in [13], after the transformation $\tilde{x} = t$, $\tilde{t} = f_{xx}$, $\tilde{f}_{\tilde{x}\tilde{x}} = -f_{xt}$, $\tilde{f}_{\tilde{x}\tilde{t}} = x$, $\tilde{f}_{\tilde{t}\tilde{t}} = f_{tt}$, it takes the form (20): $\tilde{f}_{\tilde{t}\tilde{t}\tilde{t}} = \tilde{f}^2_{\tilde{x}\tilde{x}\tilde{t}} - \tilde{f}_{\tilde{x}\tilde{x}\tilde{x}}\tilde{f}_{\tilde{x}\tilde{t}\tilde{t}}$. Notice that this is not a contact transformation. Geometrically, Eqs. (20) and (24) correspond to projectively equivalent congruences. Eq. (26) was discussed in [14] and [25]. The classification of third order equations of Monge–Amperé type was given in [1].

4.1. *Veronesé variety*

Let us first recall some of the well-known properties of the Veronesé variety $V^2 \subset P^5$ thinking of P^5 as the space of 3×3 symmetric matrices Z^{ij}, $i, j = 0, 1, 2$. Veronesé variety V^2 is the variety of matrices of rank 1

$$Z = \begin{pmatrix} X^0X^0 & X^0X^1 & X^0X^2 \\ X^1X^0 & X^1X^1 & X^1X^2 \\ X^2X^0 & X^2X^1 & X^2X^2 \end{pmatrix},$$

It can be viewed as the canonical embedding $F : P^2 \to V^2 \subset P^5$ defined by

$$Z^{ij} = X^iX^j, \quad i = 0, 1, 2, \tag{30}$$

where $X^0 : X^1 : X^2$ are homogeneous coordinates in P^2. Veronesé variety coincides with the singular locus of the *cubic symmetroid* defined by the equation

$$\det Z = 0,$$

which also is the bisecant variety of V^2 consisting of symmetric matrices of rank two. Under the embedding (30) each line in P^2 is mapped onto a conic on V^2, therefore, Veronesé variety carries a 2-parameter family of conics. The projective automorphism group of V^2 coincides with the natural action of PSL_3 on P^5

$$Z \to g^{\mathrm{T}}Zg, \quad g \in PSL_3, \tag{31}$$

which obviously preserves V^2.

Below we discuss in some more detail the geometry of congruences associated with the Eqs. (20), (24)–(29).

4.2. *Geometry of the equations with no Riemann invariants*

In this subsection we discuss Eqs. (20), (24) and (25). Rewritten as systems of conservation laws, they possess no Riemann invariants, so that the corresponding focal varieties are irreducible. We explicitly demonstrate that they coincide with different non-singular projections of the Veronesé variety.

Eq. (20). The focal variety of the corresponding congruence (22) is defined by the equations

$$y^0 = \lambda, \qquad y^1 = a\lambda - b, \qquad y^2 = b\lambda - c, \qquad y^3 = c\lambda - b^2 + ac \tag{32}$$

where λ satisfies the characteristic equation

$$\lambda^3 + a\lambda^2 - 2b\lambda + c = 0. \tag{33}$$

One can verify that the three focal surfaces (32) corresponding to the three different values of λ are, in fact, "glued" together to form the algebraic variety defined in this affine chart by a system of seven cubics

$$\begin{aligned}
&(y^0)^3 + y^0y^1 - y^2 = 0, \qquad (y^2)^2 + y^3(y^0)^2 = 0, \qquad y^1y^2y^3 + y^0(y^3)^2 - (y^2)^3 = 0,\\
&y^2(y^0)^2 + y^1y^2 + y^0y^3 = 0, \qquad (y^3)^2 - y^1(y^2)^2 + y^0y^2y^3 + y^3(y^1)^2 = 0,\\
&y^0y^1y^3 - y^0(y^2)^2 - y^2y^3 = 0, \qquad y^0y^2 + y^1(y^0)^2 + (y^1)^2 + y^3 = 0.
\end{aligned} \tag{34}$$

Variety (34) coincides with the projection of the Veronesé variety $V^2 \subset P^5$

$$y^0 = \frac{Z^{02}}{Z^{22}}, \qquad y^1 = \frac{Z^{12} - Z^{00}}{Z^{22}}, \qquad y^2 = \frac{Z^{01}}{Z^{22}}, \qquad y^3 = -\frac{Z^{11}}{Z^{22}}$$

from the point

$$\begin{pmatrix} Z^{00} & 0 & 0 \\ 0 & 0 & Z^{00} \\ 0 & Z^{00} & 0 \end{pmatrix} \tag{35}$$

into P^4. Notice that this point does not belong to the bisecant variety $S(V^2)$ and hence the projection is non-singular. Let us list some of the main properties of this projection.

1. The manifold of trisecant lines of the focal variety (34) is three-dimensional.

2. For a point p on the focal variety the set of trisecants passing through p forms a planar pencil with the vertex p.

3. The intersection of the abovementioned planar pencil with the focal variety consists of the point p and a conic.

Geometry of solutions. Consider the Plücker image of the congruence (22), which is a three-dimensional submanifold M^3 of the Grassmanian $\mathbf{G}(1,4) \subset P^9$. Since the congruence (22) is linear, M^3 is an intersection of $\mathbf{G}(1,4)$ with a P^6. Moreover, M^3 is covered by a two-parameter family of lines (images of planar pencils) so that there are three lines passing through each point of M^3. Let now M^2 be a surface in M^3. Let $p \in M^2$ be a point and T_pM^2 a tangent plane to M^2 at p. Intersecting T_pM^2 with the three planes spanned by each pair of the three lines of M^3 passing through p, we obtain three characteristic directions in T_pM^2. The integral trajectories thereof foliate M^2 by three 1-parameter families of curves. Therefore, there is a characteristic 3-web invariantly defined on any surface M^2 in M^3. One can show that solutions of the system (20) are those M^2 for which the characteristic 3-web is hexagonal (has zero curvature). We refer to [5] for the necessary definitions. It would be particularly interesting to classify algebraic surfaces in M^3 carrying hexagonal characteristic webs.

Remark. The submanifold $M^3 \subset \mathbf{G}(1,4)$ can be equivalently described as the image of the mapping $P^4 \to P^6$ defined by the system of cubics (34): this mapping blows down the lines of the congruence (22), so that the image is indeed three-dimensional.

Eq. (24). Rewritten as a system of conservation laws,

$$a_t = b_x, \qquad b_t = c_x, \qquad c_t = \big((1+bc)/a\big)_x, \tag{36}$$

this equation is associated with the congruence

$$y^1 = ay^0 - b, \qquad y^2 = by^0 - c, \qquad y^3 = cy^0 - (1+bc)/a, \tag{37}$$

the focal variety of which is defined by

$$y^0=\lambda,\qquad y^1=a\lambda-b,\qquad y^2=b\lambda-c,\qquad y^3=c\lambda-(1+bc)/a \tag{38}$$

where λ satisfies the characteristic equation

$$\lambda^3-\frac{b}{a}\lambda^2-\frac{c}{a}\lambda+\frac{1+bc}{a^2}=0. \tag{39}$$

One can verify that the three focal surfaces (38) corresponding to the three different values of λ are glued together to form the algebraic variety defined in this affine chart by a system of cubics

$$\begin{aligned}
&1+y^0\big(y^1\big)^2+y^1y^2=0,\qquad y^1\big(y^0\big)^2-y^3=0,\qquad \big(y^0\big)^3+y^0y^2y^3+\big(y^3\big)^2=0,\\
&y^0+y^0y^1y^2+y^1y^3=0,\qquad y^0y^3-y^2\big(y^0\big)^2+y^1\big(y^3\big)^2-y^3\big(y^2\big)^2=0,\\
&\big(y^0\big)^2+y^0y^1y^3+y^2y^3=0,\qquad y^0y^1+\big(y^1\big)^2y^3-y^2-y^1\big(y^2\big)^2=0.
\end{aligned} \tag{40}$$

This algebraic variety is the projection of the Veronesé variety

$$y^0=-\frac{Z^{02}}{Z^{12}},\qquad y^1=-\frac{Z^{11}}{Z^{12}},\qquad y^2=\frac{Z^{22}-Z^{01}}{Z^{12}},\qquad y^3=-\frac{Z^{00}}{Z^{12}}$$

from the point

$$\begin{pmatrix} 0 & Z^{01} & 0\\ Z^{01} & 0 & 0\\ 0 & 0 & Z^{01}\end{pmatrix} \tag{41}$$

into P^4. Notice that the two points (35) and (41) are equivalent under the action of the group (31) preserving the Veronesé variety (indeed, both matrices have the same Lorentzian signature). Hence, both projections and the corresponding congruences of trisecants are projectively equivalent. To be explicit, the projective transformation

$$y^0=-\frac{1}{Y^1},\qquad y^1=\frac{Y^2}{Y^1},\qquad y^2=\frac{Y^0}{Y^1},\qquad y^3=-\frac{Y^3}{Y^1} \tag{42}$$

identifies the systems of cubics (34) and (40). Applying this transformation to the congruence (22) and introducing the new parameters $A=-1/c$, $B=b/c$, $C=a-b^2/c$, we readily rewrite (22) in the form

$$Y^1=AY^0-B,\qquad Y^2=BY^0-C,\qquad Y^3=CY^0-(1+BC)/A$$

which coincides with (37). This gives geometric explanation of the transformation between Eqs. (20) and (24) mentioned in the introduction. On the level of systems of conservation laws (21) and (36), this transformation is a reciprocal equivalence.

Eq. (25). Rewritten as a system of conservation laws,

$$a_t=b_x,\qquad b_t=c_x,\qquad c_t=\big((c^2+b^2-ac-1)/b\big)_x, \tag{43}$$

this equation is associated with the congruence

$$y^1=ay^0-b,\qquad y^2=by^0-c,\qquad y^3=cy^0-\big((c^2+b^2-ac-1)/b\big) \tag{44}$$

whose focal surfaces are glued together to form the algebraic variety which is the projection of the Veronesé variety

$$y^0 = -\frac{Z^{02}}{Z^{12}}, \qquad y^1 = \frac{Z^{00} - Z^{22}}{Z^{12}}, \qquad y^2 = \frac{Z^{01}}{Z^{12}}, \qquad y^3 = \frac{Z^{11} - Z^{22}}{Z^{12}},$$

from the point

$$\begin{pmatrix} Z^{00} & 0 & 0 \\ 0 & Z^{00} & 0 \\ 0 & 0 & Z^{00} \end{pmatrix} \tag{45}$$

into P^4. Notice that this point is not equivalent (over the reals) to the points (35) and (41) under the action of the group (31) (indeed, the signature of (45) is Euclidean). Hence, the congruence (44) is not projectively equivalent to either of the congruences (22) or (37). The corresponding systems of conservation laws are not reciprocally related.

We point out that the Veronesé variety $V^2 \subset P^5$, being the intersection of quadrics, does not possess trisecant lines. Trisecants appear only after we project V^2 into P^4. Indeed, let P_0 be a point in P^5 not on the bisecant variety $S(V^2)$. Viewed as a 3×3 symmetric matrix, P_0 defines a non-degenerate conic in P^2

$$\sum_{i,j=0}^{2} \left(P_0^{-1}\right)^{ij} X^i X^j = 0 \tag{46}$$

where $X^0 : X^1 : X^2$ are homogeneous coordinates. If a plane passes through P_0 and cuts V^2 in three points, then pre-images of these points under the embedding (30) are pairwise conjugate with respect to the conic (46). Conversely, P_0 lies in the plane spanned by the images under (30) of any three points in P^2 which are pairwise conjugate with respect to (46). Thus, there is a three-parameter family of planes passing through P_0 and cutting V^2 in three points. Projecting this family from the point P_0 into P^4, we arrive at the congruence of lines in P^4. By a construction, its lines are trisecants of this projection, which is thus the focal surface of our congruence. To see that the developable surfaces of the congruence are planar pencils of lines, we consider a line L in P^2 defined by the equation $L_0X^0 + L_1X^1 + L_2X^2 = 0$. Under the embedding (30), this line corresponds to a conic on V^2 lying in the so called *conisecant plane* of V^2. In matrix form, equations of this plane are $LZ = 0$. The three-dimensional subspace Λ spanned by P_0 and the conisecant plane consists of all Z such that the vectors LZ and LP_0 are collinear. In addition to the conic in the conisecant plane, Λ intersects V^2 in the point P_0^L whose pre-image in P^2 under (30) has homogeneous coordinates P_0L^{T}. Consider now the one-parameter family of planes in P^5 which belong to Λ and pass through the line joining P_0 and P_0^L. Clearly, each of these planes intersects V^2 in three points, and the projection of this one-parameter family of planes into P^4 will be a planar pencil of lines. This gives developable surfaces of our congruence.

4.3. Geometry of the equations with one Riemann invariant

In this subsection we discuss Eqs. (26) and (27). Since both equations possess only one Riemann invariant, the corresponding focal varieties will be reducible, consisting of a cubic scroll and a plane, intersecting the cubic scroll along its directrix.

Eq. (26) can be rewritten as a system of conservation laws

$$a_t = b_x, \qquad b_t = c_x, \qquad c_t = (bc/a)_x \tag{47}$$

the characteristic velocities of which are $\lambda^1 = b/a$ and λ^2, $\lambda^3 = \pm\sqrt{c/a}$. The only Riemann invariant $R^1 = c/a$ corresponds to λ^1. The focal surface corresponding to λ^1 is the plane

$$y^1 = y^3 = 0, \tag{48}$$

while the focal surfaces corresponding to λ^2 and λ^3 are glued together to form the cubic scroll defined by a system of quadrics

$$y^0 y^1 + y^2 = 0, \qquad y^0 y^2 + y^3 = 0, \qquad y^1 y^3 - \left(y^2\right)^2 = 0. \tag{49}$$

The plane (48) intersects the cubic scroll along its directrix

$$y^1 = y^2 = y^3 = 0. \tag{50}$$

The cubic scroll (49) can be obtained by projecting the Veronesé variety

$$y^0 = \frac{Z^{02}}{Z^{12}}, \qquad y^1 = \frac{Z^{11}}{Z^{12}}, \qquad y^2 = -\frac{Z^{01}}{Z^{12}}, \qquad y^3 = \frac{Z^{00}}{Z^{12}},$$

from the point

$$\begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & Z^{22} \end{pmatrix}.$$

Notice that the center of this projection lies on the Veronesé variety. The directrix (50) is the image of the tangent plane $Z^{00} = Z^{01} = Z^{11} = 0$ to the Veronesé variety at the center of projection, and the plane (48) is the projection of the three-dimensional linear subspace in P^5 spanned by the tangent plane and the point

$$\begin{pmatrix} 0 & Z^{01} & 0 \\ Z^{01} & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix} \tag{51}$$

on the bisecant variety. Thus, the focal variety of our congruence is reducible and consists of the plane (48) and the cubic scroll (49). Like in the case of systems with no Riemann invariants,

1. the manifold of trisecants of the focal variety is three-dimensional, and

2. for a fixed point p on the focal variety the set of trisecants passing through p forms a planar pencil with the vertex p. If p belongs to the plane (48), the corresponding planar pencil cuts the focal variety in the point p and a conic. If p belongs to the cubic scroll, it cuts the focal variety in the point p and a pair of lines.

Eq. (27) can be rewritten as a system of conservation laws

$$a_t = b_x, \qquad b_t = c_x, \qquad c_t = \left((c^2 + b^2 - ac)/b\right)_x \tag{52}$$

the characteristic velocities of which are $\lambda^1 = c/b$ and λ^2, $\lambda^3 = (c - a \pm \sqrt{4b^2 + (c-a)^2})/2b$. The only Riemann invariant $R^1 = (c-a)/b$ corresponds to λ^1. The focal surface corresponding to λ^1 is the plane

$$y^1 = y^3, \qquad y^2 = 0, \tag{53}$$

while the focal surfaces corresponding to λ^2 and λ^3 are glued together to form the cubic scroll defined by a system of quadrics

$$y^0y^3 + y^2 = 0, \qquad y^0y^2 + y^1 = 0, \qquad y^1y^3 - (y^2)^2 = 0. \tag{54}$$

The plane (53) intersects the cubic scroll (54) along its directrix

$$y^1 = y^2 = y^3 = 0. \tag{55}$$

The cubic scroll (54) can be obtained by projecting V^2

$$y^0 = \frac{Z^{02}}{Z^{12}}, \qquad y^1 = \frac{Z^{00}}{Z^{12}}, \qquad y^2 = -\frac{Z^{01}}{Z^{12}}, \qquad y^3 = \frac{Z^{11}}{Z^{12}},$$

from the point

$$\begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & Z^{22} \end{pmatrix}$$

on V^2. The directrix (55) is the image of the tangent plane $Z^{00} = Z^{01} = Z^{11} = 0$ to V^2 at the center of projection, while the plane (53) is the image of the three-dimensional linear subspace spanned by the tangent plane and the point

$$\begin{pmatrix} Z^{00} & 0 & 0 \\ 0 & Z^{00} & 0 \\ 0 & 0 & 0 \end{pmatrix} \tag{56}$$

on the bisecant variety.

A coordinate-free construction of the congruences discussed above can be described as follows. Take a point $P_0 \in S(V^2)$ which is represented by a symmetric matrix of rank two. Then there is a nonzero vector $X \in P^2$ such that $P_0X = 0$. Consider the tangent plane to V^2 at the point $F(X)$. The projection of V^2 into P^4 from the point $F(X)$ is a cubic scroll. The projection of the tangent plane is the directrix. The projection of the three-dimensional space spanned by the tangent plane and P_0 is the plane intersecting the cubic scroll along its directrix.

Although the last two examples look pretty similar, they are not projectively equivalent over the reals, indeed, the points (51) and (56) have different signatures.

4.4. *Geometry of the equation with two Riemann invariants*

In this subsection we discuss Eq. (28). Due to the existence of two Riemann invariants, the corresponding focal variety will be reducible, consisting of two planes and a two-dimensional quadric.

Eq (28) can be rewritten as a system of conservation laws

$$a_t = b_x, \qquad b_t = c_x, \qquad c_t = \big((c^2 - 1)/b\big)_x \tag{57}$$

with the characteristic velocities $\lambda^1 = 0$ and $\lambda^2, \lambda^3 = (c \mp 1)/b$. The system has two Riemann invariants $(c \pm 1)/b$ corresponding to λ^2 and λ^3, respectively. The focal surfaces of the associated congruence corresponding to λ^2 and λ^3 are the planes

$$\begin{aligned} &y^2 = \mp 1, \\ &y^0 = \mp y^3, \end{aligned} \tag{58}$$

while the third focal surface, corresponding to λ^1, is the quadric

$$y^0 = 0, \qquad y^1 y^3 - \left(y^2\right)^2 + 1 = 0. \tag{59}$$

The planes (58) intersect the quadric (59) along the rectilinear generators

$$y^0 = 0, \qquad y^2 = \mp 1, \qquad y^3 = 0$$

which belong to different families and meet at infinity.

One can describe this congruence in a coordinate-free form as follows. Consider a quadric Q in a hyperplane $\Lambda \subset P^4$. Choose a point $p \in Q$ and draw two rectilinear generators l_1, l_2 of Q through p. Choose two planes π_1 and π_2 which are not in Λ such that $l_i \subset \pi_i$ and $\pi_1 \cap \pi_2 = p$. The union of $\pi_1,\ \pi_2$ and Q is the focal variety in question. Its trisecants define a congruence in P^4.

4.5. Geometry of the equation with three Riemann invariants

As demonstrated in [4], the focal varieties of the congruences corresponding to diagonalizable n-component T-systems, are collections of n linear subspaces of codimension two in P^{n+1}. For $n = 4$ we have 3 planes in P^4. To ensure nondegeneracy of the congruence in the sense of [4] (i.e., the lines of the congruence sweep out P^4), we require that the points of their pairwise intersections are distinct.

Eq. (29) can be rewritten as a linear system of conservation laws

$$a_t = b_x, \qquad b_t = c_x, \qquad c_t = b_x \tag{60}$$

with the characteristic velocities $\lambda^1 = 0$, $\lambda^2 = 1$, $\lambda^3 = -1$. Being linear, this system has 3 Riemann invariants. The focal surfaces of the associated congruence are the planes

$$\begin{aligned} &y^1 = y^3, \quad && y^0 = 0 \quad && \text{for } \lambda^1 = 0,\\ &y^3 = -y^2, \quad && y^0 = 1 \quad && \text{for } \lambda^2 = 1 \end{aligned}$$

and

$$y^3 = y^2, \quad y^0 = -1 \quad \text{for } \lambda^1 = 0,$$

respectively.

We would like to conclude this section by formulating two conjectures about the structure of congruences in P^{n+1} whose developable surfaces are planar pencils of lines.

Conjecture 1. *The focal varieties of such congruences are algebraic* (*possibly, reducible and singular*).

Conjecture 2. *The intersection of the focal variety with a developable surface* (*which is a planar pencil of lines*) *consists of a point* (*the vertex of the pencil*) *and a plane curve of degree* $n - 1$.

For $n = 2$ this is obvious. For $n = 3$ it follows from the classification results presented above. Both conjectures are true for general linear congruences in P^{n+1} (see Section 5) and congruences arising from the completely exceptional Monge–Ampère type equations (Section 6). This shows that the problem in question is actually algebro-geometric.

5. Linear congruences

A wide class of T-systems is provided by (general) linear congruences, i.e., the congruences (2) whose Plücker coordinates $1, u^i, f^i, u^i f^j - u^j f^i$ satisfy n linear equations of the form (23). We emphasize that all examples discussed so far belong to this class.

Let y be a fixed point in P^{n+1}. For the lines of the congruence (2) passing through q we have $f^i = u^i y^0 - y^i$, which, upon the substitution into (23), implies a linear system for u. In general, this system possesses a unique solution, so that there exists a unique line of our congruence passing through y (such congruences are said to be of order one). The focal variety V (also called the jump locus) consists of those y for which the corresponding linear system is not uniquely solvable for u. One can show that V has codimension at least two, and in the case it equals two, the developable surfaces are planar pencils of lines. Moreover, the intersection of any of these planes with the focal variety V consists of a point and a plane curve of degree $n-1$. Thus, there is a T-system associated with any linear congruence in P^{n+1}.

In the case $n = 4$ the geometry of focal varieties of general linear congruences, known as the Palatini scrolls, was investigated in [23] (see also [18] and [22] for further properties of the Palatini scrolls).

6. Completely exceptional Monge–Ampère type equations

Another important class of T-systems is provided by completely exceptional Monge–Ampère equations studied in [6]. Equations of this type are defined as follows. Introduce the Hankel matrix

$$\begin{vmatrix} \frac{\partial^{2m}u}{\partial x^{2m}} & \frac{\partial^{2m}u}{\partial x^{2m-1}\partial t} & \frac{\partial^{2m}u}{\partial x^{2m-2}\partial t^2} & \cdots & \frac{\partial^{2m}u}{\partial x^{m}\partial t^{m}} \\ \frac{\partial^{2m}u}{\partial x^{2m-1}\partial t} & \frac{\partial^{2m}u}{\partial x^{2m-2}\partial t^2} & \frac{\partial^{2m}u}{\partial x^{2m-3}\partial t^3} & \cdots & \frac{\partial^{2m}u}{\partial x^{m-1}\partial t^{m+1}} \\ \cdots & \cdots & \cdots & \cdots & \cdots \\ \frac{\partial^{2m}u}{\partial x^{m}\partial t^{m}} & \frac{\partial^{2m}u}{\partial x^{m-1}\partial t^{m+1}} & \frac{\partial^{2m}u}{\partial x^{m-2}\partial t^{m+2}} & \cdots & \frac{\partial^{2m}u}{\partial t^{2m}} \end{vmatrix} \tag{61}$$

and denote by $M_{J,K}(u)$ its minor of order l whose rows and columns are encoded in the multiindices $J = (j_1, \dots, j_l)$ and $K = (k_1, \dots, k_l)$, respectively. PDE's in question are defined by linear combinations of these minors,

$$\sum A^{J,K} M_{J,K} = 0 \tag{62}$$

where the summation is over all possible l, J, K, and $A^{J,K}$ are arbitrary constants. Any such equation can be rewritten as $\frac{\partial^{2m}u}{\partial t^{2m}} = f(\frac{\partial^{2m}u}{\partial x^{2m}}, \frac{\partial^{2m}u}{\partial x^{2m-1}\partial t}, \dots, \frac{\partial^{2m}u}{\partial x^{1}\partial t^{2m-1}})$, and after the introduction of $a^1 = \frac{\partial^{2m}u}{\partial x^{2m}}$, $a^2 = \frac{\partial^{2m}u}{\partial x^{2m-1}\partial t}, \dots, a^{2m} = \frac{\partial^{2m}u}{\partial x^{1}\partial t^{2m-1}}$, assumes the conservative form

$$a^1_t = a^2_x, \quad a^2_t = a^3_x, \dots, a^{2m}_t = f\left(a^1, a^2, \dots, a^{2m}\right)_x. \tag{63}$$

One can show that this is always a T-system (in fact, its linear degeneracy was demonstrated in [6]), and the corresponding congruence (2) has the following properties:

– its developable surfaces are planar pencils of lines,

– its focal variety has codimension at least 2,
– each developable surface intersects the focal variety in a point, which is the vertex of the pencil, and a plane curve of degree $n-1$.

To obtain systems of this type for odd n, one should consider Eq. (62) which are independent of $\frac{\partial^{2m}u}{\partial t^{2m}}$. Introducing $v=\frac{\partial u}{\partial x}$ and rewriting the resulting equation for v (which is of the order $2m-1$) as a system of conservation laws, one arrives at the congruence (2) with the properties as formulated above. We are planning to investigate the geometry of these examples elsewhere. When $n\geqslant 4$ these congruences are not necessarily linear. In this case the focal varieties must be singular, as follows from [18].

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Appendix A

We need the following lemma to prove Theorem 2.

Lemma A.1. *Let $B(\mu)$ be $2\times m$ matrix, whose entries are linear forms in $(\mu_1:\mu_2:\ldots:\mu_m)\in CP^{m-1}$, i.e., $b_{ij}(\mu)=b^k_{ij}\mu_k$. Then there exists $\tilde\mu\in CP^{m-1}$ such that* $\operatorname{rank} B(\tilde\mu)=1$.

Proof. Equation $\det|\nu_1 b^i_{1k}-\nu_2 b^i_{2k}|=0$ has at least one root $(\tilde\nu_1:\tilde\nu_2)\in CP^1$. Therefore, the system of linear equations $\sum_i(\tilde\nu_1 b^i_{1k}-\tilde\nu_2 b^i_{2k})\mu_i=0$ has a nontrivial solution $\tilde\mu=(\mu_1(\tilde\nu),\ldots,\mu_m(\tilde\nu))$. □

Proof of Theorem 2. We only need to prove that conditions 1) and 2) are sufficient (the necessity follows from the discussion in Section 3). Let $L\subset V^{n+2}$ be an n-dimensional subspace which is Lagrangian for all $\{,\}_\mu$. Choose a basis $e_1,e_2,\ldots,e_{n+2}$ in V^{n+2} such that the last n vectors $e_3,\ldots,e_n$ constitute a basis of L. In this basis,

$$A^\alpha=\begin{pmatrix} C^\alpha & B^\alpha \\ -(B^\alpha)^{\mathrm T} & 0\end{pmatrix} \tag{A.1}$$

where B^α is a $2\times n$ matrix and C^α is a skew-symmetric 2×2 matrix.

Consider $n-1$ linear equations for $\xi\in L$:

$$\{e_1,\xi\}_\alpha=0, \tag{A.2}$$

where $\{,\}_\alpha$ is the skew-symmetric scalar product defined by A^α. There exists a nonzero solution of this system. Choose this solution to be the basis vector e_3. In this basis $A^\alpha_{1,3}=0$. There must exist α for which $A^\alpha_{2,3}\neq 0$. Otherwise Lemma A.1 implies that there is μ such that $\operatorname{rank} B(\mu)=1$, which means $\operatorname{rank} A(\mu)=2$. Choose the matrix A^α for which $A^\alpha_{2,3}\neq 0$ to be A^1. Normalize A^1 so that $A^1_{2,3}=-1$ and for $\alpha=2,\ldots,n-1$ replace A^α by $A^\alpha+A^\alpha_{2,3}A^1$. Now all matrices A^α with $\alpha=2,\ldots,n-1$ have zero first column in B^α. Applying the same procedure to the linear span of $\{e_4,\ldots,e_{n+2}\}$ with $\alpha=2,\ldots,n-1$, one ends up with B^2 of the form

$$\begin{pmatrix} 0 & 0 & \dots \\ 0 & -1 & \dots \end{pmatrix}$$

and A^α for $\alpha = 3, \dots, n-1$ having two first zero columns in B^α. After $n-1$ steps, matrices A^α will have the following form: $\alpha - 1$ first columns of B^α are zero and the αth column is $(0, -1)^{\mathrm{T}}$. In particular, B^{n-1} is of the form

$$\begin{pmatrix} 0 & \dots & 0 & 0 & b^{n-1} \\ 0 & \dots & 0 & -1 & c \end{pmatrix}$$

with $b^{n-1} \neq 0$. Note that up to now only $e_3, \dots, e_{n+1}$ were fixed. Replacing e_{n+2} by $e_{n+2} + ce_{n+1}$, one gets $\tilde{c} = 0$ and $\tilde{b}_{n-1} = b_{n-1} \neq 0$.

After replacing A^{n-2} by $A^{n-2} - \frac{A^{n-2}_{1,n+2}}{b_{n-1}} A^{n-1}$, it takes the form

$$\begin{pmatrix} 0 & \dots & 0 & 0 & b^{n-2} & 0 \\ 0 & \dots & 0 & -1 & c & r \end{pmatrix}.$$

Replacing $e_{n+2} \to e_{n+2} + re_n$, $e_{n+1} \to e_{n+1} + ce_n$ does not change B^{n-1} and kills c and r. Repeating this "backward" procedure one transforms all matrices A^α to the following form: all columns of B^α are zero except for αth, which is $(0, -1)^{\mathrm{T}}$, and $(\alpha + 1)$st, which is $(b^\alpha, 0)^{\mathrm{T}}$ with $b^\alpha \neq 0$.

The above transformations of V^{n+2} induce reciprocal transformations of the associated system of conservation laws. As a result, the r.h.s. of the system assume the form

$$f^1 = b^1 u^2 + c^1, \;\; f^2 = b^2 u^3 + c^2, \dots, f^{n-1} = b^{n-1} u^{n-2} + c^{n-1},$$

where the constants c^α are the nonzero elements of the corresponding 2×2 matrices C^α in (A.1). Finally, the renormalization $u^i \to d^i u^i$ with $d^i = \prod_i^{n-1} b^k$ completes the proof, since the constants c^k do not effect (1). □

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